Distributed almost exact approximations for minor-closed families

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Abstract. We give efficient deterministic distributed algorithms which given a graph G from a proper minor-closed family $\mathcal{C}$ find an approximation of a minimum dominating set in G and a minimum connected dominating set in G. The algorithms are deterministic and run in a polylogarithmic number of rounds. The approximation accomplished differs from an optimal by a multiplicative factor of $(1 + o(1))$.

1 Introduction

The most fundamental challenge in theory of distributed algorithms is to determine how the local structure of a network impacts its global properties. This leads to a completely different computational paradigm than the sequential model or the massively parallel PRAM model. Not surprisingly, many problems which admit efficient sequential protocols, such as the maximum matching problem or the maximal independent set problem elude efficient distributed solutions. In this paper, we will study distributed approximations for two classical graph-theoretic problems assuming the underlying graph belongs to a proper minor-closed family. We will consider the distributed model which was introduced by Linial in [L92]. In this model, the network is represented by an undirected graph with vertices corresponding to processors, and edges corresponding to communication links between processors. The network is synchronized and computations proceed in discrete rounds. In a single round a vertex can send and receive messages from its neighbors, and can perform some local computations. Neither the amount of local computations nor the lengths of messages is restricted in any way. Importantly, we will also assume that nodes in the network have unique identifiers which are positive integers from $\{1, \ldots, n\}$ where $n = poly(|G|)$ is globally known and $|G|$ is the order of the graph.

1.1 Results

Although different possible measures of efficiency of a distributed algorithm can be considered, traditionally a deterministic distributed algorithm is called *efficient* in the model if it runs in a poly-logarithmic (in the order of the graph) number of rounds. Only very few classical graph-theoretic problems are known to admit an efficient deterministic distributed algorithm. For example, even the maximal independent set problem, for which an efficient deterministic PRAM algorithm exists [L86], still has an unknown distributed complexity. In this paper, we shall focus on distributed approximation algorithms for two classical problems, the minimum dominating set problem and the minimum connected dominating set problem. Let β be a graph-theoretic function to be optimized and let β^* denote its optimal value. An *almost exact approximation* for the optimization problem is a distributed approximation algorithm which given a positive integer k, finds in a graph G in a poly-logarithmic number of rounds a solution with value of at least $(1 - O(1/\ln^k |G|))\beta^*(G)$, where $|G|$ is the order of G. For example, Kuhn et. al. in [KMNW05b] give almost-exact approximations for the maximum independent set and minimum dominating set problems in unit-disk graphs.

In this paper we will give efficient distributed approximation algorithms for the minimum dominating set problem and the minimum connected dominating set problem for graphs which are from a proper minor-closed family. Let $G = (V, E)$ be a graph. Graph H is called a minor of G if for some subgraph G' of G, there is a partition of $V(G')$ into $V_1, \dots, V_l$, such that the graph $\bar{H}$, with vertex set $\{1, \dots, l\}$ and edges between i and j whenever there is an edge in G' with one endpoint in V_i, another in V_j, is isomorphic to H. It is well-known (see [D97]) that H is a minor of G if and only if it can be obtained from a subgraph of G be a series of edge contractions. An infinite family of graphs $\mathcal{C}$ is called minor-closed when for every graph $G \in \mathcal{C}$ any minor of G is also in $\mathcal{C}$. A family $\mathcal{C}$ is called proper if there exists a graph which is not in $\mathcal{C}$, i.e. $\mathcal{C}$ is not the family of all graphs. Certainly, the most important example of a proper minor-closed family is the class of planar graphs. For $\mathcal{C}$, let $\rho_{\mathcal{C}}$ be the infimum of the edge density of graphs from $\mathcal{C}$. Complexity of algorithms depends on $\rho_{\mathcal{C}}$ and we will often use the fact that if $\mathcal{C}$ is proper then $\rho_{\mathcal{C}}$ is finite (see [NM05]).

Distributed approximation algorithms for planar graphs were studied in [CH04] and [CHS06]. In [CH04], almost exact approximation are obtained for the maximum-weight independent set problem provided the underlying graph is planar. In [CHS06], an almost exact approximation for the maximum matching problem is given in planar graphs and an almost exact approximation for the minimum dominating set problem is given in planar graphs that do not contain $K_{2,\ln |G|}$ as a subgraph. In this paper we will not only get rid of the annoying additional assumption on planar graphs from [CHS06] but also we will show how to solve the problems in any minor closed family $\mathcal{C}$. Finally, we will prove that the minimum connected dominating set problem can be approached in a very similar way.

A *dominating set* in a graph G is a subset D of vertices such that for every vertex $v \notin D$ a neighbor u of v belongs to D. By $\gamma(G)$ we will denote the cardinality of a smallest dominating set in G. A dominating set D is called a *connected dominating set* in G if in addition, the subgraph of G induced by D is connected. We will denote by $\gamma_c(G)$ the cardinality of the smallest connected dominating set in a connected graph G. For the minimum dominating set problem, we will prove that there is a distributed algorithm which given positive integer q finds in graph $G \in \mathcal{C}$ a dominating set $\bar{D}$ such that $|\bar{D}| \leq \left(1 + O\left(\frac{1}{\ln^q |G|}\right)\right)\gamma(G)$. The algorithm runs in $O(\ln\ln|G| \ln^*|G| \ln^{1+r}|G|)$ rounds where $r = 6(q+1)\rho_{\mathcal{C}} \ln 3$. (Theorem 1.) For the minimum connected dominating set problem we will show that there is a distributed algorithm which finds in a connected graph $G \in \mathcal{C}$ a connected dominating set $\bar{D}$ such that $|\bar{D}| \leq \left(1 + O\left(\frac{1}{\ln^q |G|}\right)\right)\gamma_c(G)$. The algorithm runs in $O(\ln\ln|G| \ln^*|G| \ln^{1+r}|G|)$ rounds with $r = 6(q+1)\rho_{\mathcal{C}} \ln 3$. (Theorem 2.)

1.2 Related Work

We briefly indicate how our contribution compares with other results refering to Elkin's survey [E04], for a more comprehensive overview. First let us mention that efficient distributed algorithms that find an exact solution do not exist for the minimum dominating set problem even when restricted to cycles [L92]. In addition, recently, Kuhn et. al. in [KMW04] showed that the number of rounds required to achieve a poly-logarithmic approximation ratio for minimum dominating set is at least $\Omega(\sqrt{\log|G|/\log\log|G|})$ or $\Omega(\log\Delta/\log\log\Delta)$, where Δ denotes the maximum degree of graph G.

On a more positive note, Kutten and Peleg [KP95] gave an efficient distributed algorithm which finds a dominating set of size at most $|G|/2$ in an arbitrary graph G. Not surprisingly, if randomization is allowed, then fast approximations can be obtained. In particular, a nice algorithm from [KW03] gives a randomized $O(k\Delta^{2/k}\log\Delta)$-approximation in a constant time using an LP relaxation. As in the case of the minimum dominating set problem, efficient randomized algorithms for the connected dominating set are known [DPRS03].

It is also worth mentioning that our algorithms share many similarities with almost-exact approximations for the above problems in unit-disk graphs from [KMNW05b] and particularly [CH06]. Specifically, algorithms for unit-disk graphs and graphs from a minor closed families are both attacked by first finding a cluster graph and then perform computations locally. Clustering from [CH04] (as well as [KMNW05b]) exploits the bounded-growth property of unit-disk graphs and is based on the ruling-set technique from [AGLP89]. The clustering in this paper, generalizes the clustering procedures from [CH04] and [CHS06] and relies on properties of minor-monotone families.

1.3 Notation and Organization

We will use the standard graph-theoretic notation and terminology. In particular, following the convention from [D97], for a graph G, $|G|$ will denote the number of vertices in G and $||G||$ the number of edges. In the rest of the paper we will first give an auxiliary distributed $O(\ln|G|)$-approximation for the minimum dominating set problem in a graph $G \in \mathcal{C}$ (Section 2). Section 2 also contains a generalization of the clustering from [CH04] to minor-closed families $\mathcal{C}$. In Section 3, we give our approximation algorithms and give a specification to the important case when $\mathcal{C}$ is the class of planar graphs.

2 Tools

Let $\mathcal{C}$ be a proper minor-closed family of graphs. In this section, we will describe two auxiliary algorithms. The first procedure finds a $O(\ln|G|)$-approximation of the minimum dominating set in a graph $G \in \mathcal{C}$. This is a very simple distributed greedy algorithm which will be used as an initial procedure that yields an auxiliary graph which is further clustered by the main algorithm. The second procedure is a modification of the clustering algorithm from [CH04]. This is our main tool for finding a clustering of a graph from $\mathcal{C}$.

2.1 Distributed $O(\ln|G|)$-approximation

For a proper minor-closed family $\mathcal{C}$ let $\rho_{\mathcal{C}}$ be the edge density of $\mathcal{C}$, i.e. $\rho_{\mathcal{C}}$ is the infimum of ρ such that for every graph $G \in \mathcal{C}$, $||G|| \leq \rho|G|$. Then $\rho_{\mathcal{C}}$ is finite as long as $\mathcal{C}$ is proper (see [NM05]) and if G is nontrivial (i.e. contains a nonempty graph) then $\rho_{\mathcal{C}} \geq 0.5$. Let $G \in \mathcal{C}$ and suppose that V_1, V_2 is a partition of V. Let

$$deg_i(v) = |N(v) \cap V_i|, \Delta_i = \max_v deg_i(v)$$

where $N(v)$ is a set of neighbours of v. In addition for $S \subset V$ let $N_i(S)$ denote the set of vertices in V_i which have a neighbor in S.

Lemma 1. *Let $\mathcal{C}$ be a proper nontrivial minor closed family. Let G be a nonempty graph from $\mathcal{C}$, let V_1, V_2 be a partition of $V(G)$ and let $B = \{v | deg_1(v) \geq \Delta_1/2\}$. If $\Delta_1 \geq 4\rho_{\mathcal{C}}$ and D is a subset of V which dominates all vertices from V_1 then*

$$|D| \geq \frac{|B|}{6\rho_{\mathcal{C}}(2\rho_{\mathcal{C}}+1)}.$$

We will consider the following greedy algorithm.

GreedyDS
Input: Graph $G = (V, E)$ from $\mathcal{C}$.
Output: Dominating set D^* in G.

(1) $D^* := \emptyset$, $V_1 := V$, $V_2 := \emptyset$.

(2) for $i := 0$ to $\lfloor \lg |G| \rfloor - \lceil \lg 4\rho_{\mathcal{C}} \rceil - 1$ do
  (a) Let $B := \{v | deg_1(v) \geq |G|/2^{i+1}\}$.
  (b) If $v \in V_1$ and $N(v) \cap B \neq \emptyset$ then move v from V_1 to V_2.
  (c) $D^* := D^* \cup B$. Delete all vertices in B and all edges incident to B from G.
(3) Let $D^* := D^* \cup V_1$. Return D^*.

We shall first make a few preliminary observations about GreedyDS. Let $G^{(i)}$ be the graph in the ith iteration of the for loop. Similarly let $V_k^{(i)}$ ($B^{(i)}$), be the set V_k (B) in the ith iteration and let $\Delta_1^{(i)} = \Delta_1(G^{(i)})$. We first observe the following easy lemma.

Lemma 2. *Let $\mathcal{C}$ be a proper nontrivial minor closed family and let $G \in \mathcal{C}$.*

- *We have $\Delta_1^{(i)} \leq |G|/2^i$.*
- *If $B^{(i)} \neq \emptyset$ then $\Delta_1^{(i)} \geq 4\rho_{\mathcal{C}}$.*

We can now prove the main property of GreedyDS.

Lemma 3. *Let $\mathcal{C}$ be a proper nontrivial minor closed family. Let D be a dominating set in graph $G = (V, E)$ from $\mathcal{C}$. Then*

$$|B^{(i)}| \leq 6\rho_{\mathcal{C}}(2\rho_{\mathcal{C}} + 1)|D|.$$

In addition, if V_1^* denotes the set of vertices in V_1 in the step (3) of GreedyDS *then*

$$|V_1^*| \leq (4\rho_{\mathcal{C}} + 2)|D|.$$

Proof. Let $B^{(<i)} := B^{(0)} \cup \cdots \cup B^{(i-1)}$. Vertices from $V_1^{(i)}$ cannot be dominated by vertices from $B^{(<i)}$ as all neighbors of $B^{(<i)}$ in G are contained in $B^{(<i)} \cup V_2^{(i)}$. Consequently $D \cap (V_1^{(i)} \cup V_2^{(i)})$ is a set which dominates $V_1^{(i)}$ in $G^{(i)}$. By Lemma 2, if $B^{(i)} \neq \emptyset$ then $\Delta_1(G^{(i)}) \geq 4\rho_{\mathcal{C}}$ and we have $B^{(i)} \subseteq \{v | deg_1(v) \geq \Delta_1^{(i)}/2\}$. As $G^{(i)}$ is a subgraph of G, Lemma 1 implies that

$$|B^{(i)}| \leq 6\rho_{\mathcal{C}}(2\rho_{\mathcal{C}} + 1)|D \cap (V_1^{(i)} \cup V_2^{(i)})| \leq 6\rho_{\mathcal{C}}(2\rho_{\mathcal{C}} + 1)|D|.$$

To prove the second part, note that after the iterations from step (2), the maximum degree $\Delta_1 \leq 4\rho_{\mathcal{C}} + 2$. As a result, to dominate all vertices from V_1^* at least $|V_1^*|/(4\rho_{\mathcal{C}} + 2)$ vertices are needed.

Lemma 4. *Let $\mathcal{C}$ be a minor closed family with $\rho_{\mathcal{C}} > 0$ and let $G \in \mathcal{C}$.* GreedyDS *finds a dominating set D^* with*

$$|D^*| = O(\ln |G| \gamma(G)),$$

where $\gamma(G)$ is the size of the minimum dominating set in G.

Proof. D^* is a dominating set as in step (3) all of the remaining vertices from V_1 are added to D^*. There are less than $\lg |G| = \Theta(\ln |G|)$ iteration of step (2) and so, by Lemma 3, $|D^*| = O(\ln |G| \gamma(G))$.

2.2 Clustering algorithm

We will modify the clustering method from [CHS06] (see also [CH04]) which was applied there to planar graphs. The basic idea of the method is to find appropriate subgraphs of a graph and contract them. The process is repeated $O(\ln \ln |G|)$ times and the vertices of the graph obtained from contractions in all of the previous iterations give clusters of G. To find appropriate subgraphs it is neccessary to consider weights on edges. We shall start with the following basic observation.

Lemma 5. *Let $\mathcal{C}$ be a proper minor closed family. Let $G = (V, E)$ be a graph from $\mathcal{C}$ and let $A = \{v | deg(v) \leq 3\rho_{\mathcal{C}}\}$. Then*

$$|A| \geq |G|/3.$$

As mentioned before, we will assumme that vertices have unique identifiers which are positive integers. For $v \in V(G)$ the identifier of v will be denoted by $ID(v)$. Note that if $ID(v) \leq n$ for every vertex from $V(G)$ then $|G| \leq n$.

Decomposition
Input: $G \in \mathcal{C}$, number n such that $ID(v) \leq n$ for $v \in V(G)$.
Output: Partition $V_1, \ldots, V_{\log_k n}$ of G with $k = O(1)$.

1. Let $U := V(G)$, $i := 1$ and $k := (9\rho_{\mathcal{C}} + 3)/(9\rho_{\mathcal{C}} + 2)$.
2. Iterate $\log_k n + 1$ times:
   (a) Let A be the set of vertices in $G[U]$ of degree at most $3\rho_{\mathcal{C}}$.
   (b) Use the Cole-Vishkin algorithm from [CV86] to find a maximal independent set I in the subgraph of $G[U]$ induced by A.
   (c) $V_i := I$, $i := i + 1$, $U := U \setminus I$.

Lemma 6. *[CH04] Let $G = (V, E)$ be a graph from $\mathcal{C}$ such that the identifiers of V are in $\{1, \ldots, n\}$. Then the procedure* Decomposition *finds a partition $V_1, \ldots, V_{\log_k n}$ of $V(G)$ such that each V_i is an independent set and for every $v \in V_i$, $deg(v, \bigcup_{j>i} V_j) \leq 3\rho_{\mathcal{C}}$. The algorithm runs in $O(\ln^* n \ln n)$ rounds.*

We will now describe our clustering algorithm. This is essentially the algorithm from [CH04] which is here adopted to minor-closed families. Main idea of the algorithm is to find appropriate subgraphs of G and contract the subgraphs so that the number of contracted edges is a constant fraction of $||G||$. The process is iterated $O(\ln \ln n)$ times where $|G| \leq n$. We will identify graphs with their edge sets and if ω is a weight function defined on the edge set of graph H then for $F \subseteq E(H)$, $\omega(F) := \sum_{e \in F} \omega(e)$. In addition, $N(w)$ will denote the set of neighbors of vertex w.

Clustering
Input: Graph $G = (V, E) \in \mathcal{C}$, number n such that $ID(v) \leq n$ for every $v \in V$, positive integer c.
Output: Partition of V.

1. $H := G$ and let $\omega(e) := 1$ for every $e \in H$. Let $l := 6c\rho_{\mathcal{C}} \ln\ln n$.
2. Iterate l times:
   (a) Call DECOMPOSITION to find a partition $W_1, \ldots, W_K$ of H with $K = O(\ln n)$. Set $W_{K+1} := \emptyset$ and let $Z_i := \bigcup_{j>i} W_j$.
   (b) For every vertex w:
   (c) If i is such that $w \in W_i$ and $N(w) \cap Z_i \neq \emptyset$ then:
      – Let $u(w)$ be a vertex in $N(w) \cap Z_i$ such that
$$\omega(\{w, u(w)\}) := \max_{v \in N(w) \cap Z_i} \omega(\{w, v\}).$$
      – Add $\{w, u(w)\}$ to the auxiliary graph F.
   (d) Each connected component of F is a tree of diameter $O(K) = O(\ln n)$. For each tree T in F, in parallel, find a set of disjoint stars $S_1 \ldots S_k$ in T such that $\omega(S_1 \cup \cdots \cup S_k) \geq \omega(T - (S_1 \cup \cdots \cup S_k))$.
   (e) Modify H as follows:
      – Contract each star S_i to a new vertex $x(S_i)$.
      – For every vertex $x(S_i)$ and $y \in V(H) \cap N(S_i)$ set the weight of
$$\omega(\{x(S_i), y\}) := \sum_{u \in V(S_i) \cap N(y)} \omega(\{u, y\})$$
      and set $V(H) := \bigcup\{x(S_i)\} \cup (V(H) - \bigcup V(S_i))$.
3. If $V(H) = \{v_1, ..., v_L\}$ then for each v_i let V_i be the set of vertices of G contracted to v_i in all of the above iterations. Return $V_1, \ldots, V_L$.

Note that graph H obtained in each iteration of CLUSTERING belongs to $\mathcal{C}$ and so its edge density is at most $\rho_{\mathcal{C}}$. We can summarize the performance of CLUSTERING as follows.

Lemma 7. *Let $V_1, \ldots, V_L$ be the clusters in G obtained from* CLUSTERING. *Then*

1. *For every i, $G[V_i]$ is a subgraph of diameter $O(\ln^d n)$, where*
$$d = 6c\rho_{\mathcal{C}} \ln 3.$$
2. *The number of edges connecting different clusters is $O(||G||/\ln^c n)$.*
3. CLUSTERING *runs in $O(\ln\ln n \ln^* n \ln^{1+d} n)$ rounds.*

3 Domination Problems

Let $\mathcal{C}$ be a proper minor-closed family of graphs. In this section, we will give almost exact approximations for the minimum dominating set problem and for the connected minimum dominating set problem in graphs G such that $G \in \mathcal{C}$. Instead of finding a clustering directly in graph G, it is very convenient to work in an auxiliary graph that arises from the $O(\ln n)$-approximation of the dominating set and perform the clustering in this graph. By virtue of the minor-closed property of $\mathcal{C}$, the auxiliary graph will also be a member of $\mathcal{C}$.

3.1 Minimum Dominating Set

We will start with an almost exact approximation for the minimum dominating set problem.

Definition 1. *Let $G = (V, E)$ be a graph and let $D = \{v_1, \ldots, v_l\} \subseteq V$ be a dominating set in G. Then let $\mathbf{A}(D, G)$ be the graph $(\mathbf{V}, \mathbf{E})$ obtained as follows.*

- *Partition $V = V_1 \cup V_2 \cup \cdots \cup V_l$ so that (1) $v_i \in V_i$ and (2) for every $v \in V \setminus D$, $v \in V_i$ if $\{v, v_i\} \in E$ and if $\{v, v_j\} \in E$ for $j \neq i$ then $ID(v_j) > ID(v_i)$.*
- *$\mathbf{V} := \{V_1, \ldots, V_l\}$ (contract V_i to a vertex) and $\{V_i, V_j\} \in \mathbf{E}$ ($i \neq j$) if there is an edge in G between a vertex from V_i and a vertex from V_j.*

In addition, we will call v_i the *center* of V_i. Let $\mathbf{U}$ be a subset of $\mathbf{V}$ then $\mathbf{U}$ corresponds in a natural way to subset U of $V(G)$ by

$$U := \bigcup_{W \in \mathbf{U}} W.$$

If $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_l$ is a partition of $\mathbf{V}$ then the corresponding sequence $U_1, U_2, \ldots, U_l$, with $U_i := \bigcup_{W \in \mathbf{U}_i} W$, is a partition of $V(G)$. We will then say that $U_1, U_2, \ldots, U_l$ *arises* from $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_l$. Finally, for a graph $H = (X, F)$ if $Y \subseteq X$ then $bd(Y)$ will denote the set of all vertices in Y which have a neighbor in $X \setminus Y$.

For a dominating set D obtained by GreedyDS in G let $\mathbf{A} := \mathbf{A}(D, G)$. From Lemma 4

$$|\mathbf{A}| = O(\gamma(G) \ln |G|). \tag{1}$$

Since $\mathbf{A}$ is obtained from G be contracting V_i's, $\mathbf{A} \in \mathcal{C}$. In addition, identifiers of vertices from $\mathbf{A}$ are bounded from above by $n = poly(|G|)$.

ApproxDS
Input: Graph $G = (V, E)$ from $\mathcal{C}$ with $ID(v) \leq n$ for any $v \in V(G)$, a positive integer q.
Output: Dominating set $\bar{D}$ in G.

1. Call GreedyDS to find a dominating set D and consider $\mathbf{A} = (\mathbf{V}, \mathbf{E})$.
2. Call Clustering with $c = 1 + q$ in $\mathbf{A}$.
3. Let $\mathbf{U}_1, \ldots, \mathbf{U}_l$ be a partition of $\mathbf{V}$ and let $U_1, \ldots, U_l$ be a partition of $V(G)$ that arises from $\mathbf{U}_1, \ldots, \mathbf{U}_l$.
4. In each U_i in parallel:
   (a) Find locally in U_i a set $D_i \subseteq U_i$ of the smallest size such that D_i dominates $U_i \setminus bd(U_i)$ in G.
   (b) Let C_i be the set of centers of vertices from $bd(\mathbf{U}_i)$.
   (c) Let $\bar{D}_i := D_i \cup C_i$.
5. Return $\bar{D} := \bigcup_{i=1}^{l} \bar{D}_i$.

Theorem 1. *Let $\mathcal{C}$ be a proper minor-closed family of graphs. Algorithm* ApproxDS *finds in a graph $G \in \mathcal{C}$ a dominating set $\bar{D}$ such that*

$$|\bar{D}| \leq \left(1 + O\left(\frac{1}{\ln^q |G|}\right)\right)\gamma(G)$$

in $O(\ln \ln |G| \ln^* |G| \ln^{1+r} |G|)$ rounds where $r = 6(q+1)\rho_{\mathcal{C}} \ln 3$.

Proof. We will first note that set $\bar{D}$ returned in step five of ApproxDS is a dominating set. Indeed let $v \in V(G)$ and suppose that $v \notin \bar{D}$. If for some i, $v \in U_i \setminus bd(U_i)$ then v is dominated by a vertex from D_i. Otherwise for some i, $v \in bd(U_i)$ and so, by definition of $\mathbf{A}$, $v \in W$ for some $W \in bd(\mathbf{U}_i)$. Consequently, v is dominated by the center of W and the center is in C_i. To establish the bound for $|\bar{D}|$ let us first recall that $\mathbf{A} \in \mathcal{C}$ and so $||\mathbf{A}|| = O(|\mathbf{A}|)$. In addition the graph induced by border vertices of $\mathbf{A}$, i.e. $\mathbf{A}\left[\bigcup_i bd(\mathbf{U}_i)\right]$, has denisty $\rho_{\mathcal{C}}$ and so $|\bigcup_i bd(\mathbf{U}_i)| = O\left(||\mathbf{A}\left[\bigcup_i bd(\mathbf{U}_i)\right]||\right)$ and by Lemma 7 part 2 applied with $n = |G|$, $|\bigcup_i bd(\mathbf{U}_i)| = O(||\mathbf{A}||/\ln^{q+1}|G|)$ as $|\mathbf{A}| \leq |G|$. Consequently, as $||\mathbf{A}|| = O(|\mathbf{A}|)$ and (1) holds, $|\bigcup_i bd(\mathbf{U}_i)| = O(\gamma(G)/\ln^q |G|)$. Since $bd(\mathbf{U}_i)$ are pairwise disjoint and $|bd(\mathbf{U}_i)| = |C_i|$, we have

$$\sum_{i=1}^{l} |C_i| = O(\gamma(G)/\ln^q |G|). \tag{2}$$

Let D^* be a dominating set in G with $|D^*| = \gamma(G)$. Then $|D^* \cap U_i| \geq |D_i|$ as every vertex in $U_i \setminus bd(U_i)$ must be dominated by a vertex from $D^* \cap U_i$. Consequently $\gamma(G) = |D^*| = \sum_{i=1}^{l} |D^* \cap U_i| \geq \sum_{i=1}^{l} |D_i|$ and so $|\bar{D}| \leq \sum_{i=1}^{l} |D_i| + \sum_{i=1}^{l} |C_i| \leq \gamma(G) + \sum_{i=1}^{l} |C_i|$ which in view of (2) gives $|\bar{D}| = \left(1 + O\left(\frac{1}{\ln^q |G|}\right)\right)\gamma(G)$. To estimate the running time, note that Clustring runs in $O(\ln \ln |G| \ln^* |G| \ln^{1+r} |G|)$ rounds in $\mathbf{A}$ and every vertex in $\mathbf{A}$ has diameter of at most two in G. In addition for every i, $\mathbf{A}[\mathbf{U}_i]$ has diameter $O(\ln^r |G|)$ by Lemma 7 part 1 and so the diameter of each $G[U_i]$ is also $O(\ln^r |G|)$. Therefore, finding D_i and C_i can be done in $O(\ln^r |G|)$ rounds and the time complexity of ApproxDS is $O(\ln \ln |G| \ln^* |G| \ln^{1+r} |G|)$.

3.2 Minimum Connected Dominating Set

An algorithm for the minimum connected dominating set problem is very similar. In fact the first three steps are identical and only a very small change must be made in steps four and five. First note that the auxiliary graph $\mathbf{A}$ satisfies

$$|\mathbf{A}| = O(\gamma_c(G) \ln |G|) \tag{3}$$

where $\gamma_c(G)$ is the size if the smallest connected dominating set as $\gamma(G) \leq \gamma_c(G) \leq 3\gamma(G)$ in any connected graph G.

ApproxCDS
Input: A connected graph $G = (V, E) \in \mathcal{C}$, a positive integer q.
Output: A connected dominating set $\bar{D}$ in G.

1. Call GreedyDS to find a dominating set D and consider $\mathbf{A} = (\mathbf{V}, \mathbf{E})$.
2. Call Clustering with $c = 1 + q$ in $\mathbf{A}$.
3. Let $\mathbf{U}_1, \ldots, \mathbf{U}_l$ be a partition of $\mathbf{V}$ obtained in step 2. Let $U_1, \ldots, U_l$ be a partition of $V(G)$ that arises from $\mathbf{U}_1, \ldots, \mathbf{U}_l$.
4. In each U_i in parallel:
   (a) Let C_i be the set of centers of vertices from $bd(\mathbf{U}_i)$.
   (b) Find locally in U_i a set $D_i \subseteq U_i$ of the smallest size such that D_i dominates U_i in G, $G[D_i]$ is a connected subgraph of G, and $C_i \subseteq D_i$.
   (c) For every cluster U_j such that there is an edge in $\mathbf{A}$ between $\mathbf{U}_i$ and $\mathbf{U}_j$ find the shortest path P_{ij} between a vertex from D_i and a vertex from D_j and let $P_i := \bigcup V(P_{ij})$ where the union is taken over all of these paths.
   (d) Let $\bar{D}_i := D_i \cup P_i$.
5. Return $\bar{D} := \bigcup_{i=1}^{l} \bar{D}_i$.

The argument is slightly different than the one given for Theorem 1 as this time the main part of the argument is to show that that G contains a connected dominating set D' such that $|D'| \leq (1 + O(1/\ln^q |G|))\gamma_c(G)$, $G[D' \cap U_i]$ is a connected subgraph, $D' \cap U_i$ dominates U_i, and $C_i \subseteq D' \cap U_i$.

Lemma 8. *Let $G \in \mathcal{C}$ be a connected graph. Then G contains a connected dominating set D' such that $|D'| \leq (1+O(1/\ln^q |G|))\gamma_c(G)$ and for every $i = 1, \ldots, l$*

1. *$G[D' \cap U_i]$ is a connected subgraph of G,*
2. *$D' \cap U_i$ dominates U_i,*
3. *$C_i \subseteq D' \cap U_i$.*

Theorem 2. *Let $\mathcal{C}$ be a minor-closed family. Algorithm* ApproxDS *finds in a connected graph $G \in \mathcal{C}$ a connected dominating set $\bar{D}$ such that*

$$|\bar{D}| \leq \left(1 + O\left(\frac{1}{\ln^q |G|}\right)\right)\gamma_c(G)$$

in $O(\ln\ln |G| \ln^* |G| \ln^{1+r} |G|)$ rounds where $r = 6(q+1)\rho_{\mathcal{C}} \ln 3$.

Proof. First note that the running time can be proved in the same way as in the case of Theorem 1. Also, clearly, $\bar{D}$ is a dominating set in G. We claim that $G[\bar{D}]$ is a connected subgraph in G. Clearly $G[\bar{D}_i]$ is a connected subgraph and since G is connected so is $\mathbf{A}$. Consider the graph $\mathbf{C}(\mathbf{A})$ obtained from $\mathbf{A}$ by contracting each $\mathbf{U}_i$ to a single vertex. $\mathbf{C}(\mathbf{A})$ is clearly a connected graph. Since $C_i \subseteq D_i$, it is enough to note that whenever there is an edge $\{\mathbf{U}_i, \mathbf{U}_j\}$ in $\mathbf{C}(\mathbf{A})$ then there is a path P_{ij} in $G[\bar{D}]$ connecting a vertex from C_i with a vertex from C_j. To estimate $|\bar{D}|$ let us first show that

$$\sum_{i=1}^{l} |P_i| = O(\gamma_c(G)/\ln^q |G|). \tag{4}$$

By Lemma 7 part 2, the sum of degrees of vertices in $\mathbf{C}(\mathbf{A})$ is $O(|\mathbf{E}|/\ln^{q+1}|G|) = O(|\gamma_c(G)|/\ln^q|G|)$. Consequently, the number of P_{ij}'s is $O(|\gamma_c(G)|/\ln^q|G|)$. In addition, $|V(P_{ij})| \leq 4$ as if there is an edge $\{W, W'\} \in \mathbf{E}$ with $W \in \mathbf{U}_i$ and $W' \in \mathbf{U}_j$ then there exist $w \in W$ and $w' \in W'$ such that $\{w, w'\}$ is the edge in G. Since the center of W is in D_i and the center of W' is in D_j, the shortest path between D_i and D_j contains at most four vertices. Consequently, $\sum_{i=1}^{l}|\bar{D}_i| = \sum_{i=1}^{l}|D_i| + O(\gamma_c(G)/\ln^q|G|)$.

Finally from Lemma 8, there exists a connected dominating set D' in G such that $|D'| \leq (1+O(1/\ln^q|G|))\gamma_c(G)$, $G[D' \cap U_i]$ induces a connected subgraph, $D' \cap U_i$ dominates U_i, and $C_i \subseteq D' \cap U_i$ for every i. Since D_i, found in the step 4(b), is a set of the smallest size such that $G[D_i]$ is a connected subgraph, D_i dominates U_i and $C_i \subseteq D_i$, we must have $|D_i| \leq |D' \cap U_i|$. As a result, $|\bar{D}| = \sum_{i=1}^{l}|\bar{D}_i| = \sum_{i=1}^{l}|D_i| + O(\gamma_c(G)/\ln^q|G|) \leq \sum_{i=1}^{l}|D' \cap U_i| + O(\gamma_c(G)/\ln^q|G|) = |D'| + O(\gamma_c(G)/\ln^q|G|)$ and so $|\bar{D}| \leq \left(1 + O\left(\frac{1}{\ln^q|G|}\right)\right)\gamma_c(G)$.

3.3 Planar graphs

Class of planar graphs $\mathcal{P}$ has $\rho_{\mathcal{P}} = 3$ and so by Theorem 1 and Theorem 2 we have almost exact approximations for the minimum dominating set problem and the minimum connected dominating set problem in planar graphs that achieve the approximation error of $O(1/\ln^q|G|)$ and run in $O(\ln\ln|G|\ln^*|G|\ln^{1+r}|G|)$ rounds where $r = 18(q+1)\ln 3$. In [CHS06] an approximation algorithm for the minimum dominating set problem with $q = 1$ is given for the special subclass of planar graphs. The algorithm from [CHS06] runs in $O(\ln\ln|G|\ln^*|G|\ln^{1+r}|G|)$ rounds where $r = 27.7$ and so it is slightly faster than the algorithms from Theorem 1 and Theorem 2 which run $O(\ln\ln|G|\ln^*|G|\ln^{1+r}|G|)$ rounds where $r = 36\ln 3$. We can however apply techniques from [CHS06] and reduce the time complexity significantly. Using the SmallCluster procedure from [CHS06] and the fact that star arboricity of a planar graph is at most five, we can achieve the approximation error of $O(1/\ln^q|G|)$ in $O(\ln\ln|G|\ln^*|G|\ln^{1+r}|G|)$ rounds where $r < 5.54(q+1)$. In fact, using Tutte's Theorem on the tree arboricity of a graph with a bounded density of any subgraph (see [D97]), we can apply SmallCluster procedure and reduce the time complexity of our algorithms for minor-closed families. Due to space limitations, we will not give this refinement here.

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